Optimal Spatial Emissions

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May 17, 2024

Abstract

We present a multi-sector quantitative spatial model designed to elucidate the optimal spatial allocation of carbon emissions. We propose an emission allocative efficiency (EAE) measure that quantifies the potential for emission reallocation to increase aggregate real income. We develop a sufficient statistics approach that links EAE with data observations of inter-regional trade, labor, and carbon emissions. Based on EAE, we develop an algorithm to solve for high-dimensional optimal spatial carbon taxes. Applying our framework to the Chinese economy in 2017, we find that (i) EAE is highly correlated with the ratio of optimal carbon taxes over observed carbon taxes; (ii) optimal carbon taxes are negatively correlated with the Katz-Bonacich centrality within the observed inter-regional trade network; (iii) implementing optimal carbon taxes could increase Chinese real income by 1.42% while keeping total emissions unchanged.

JEL classification: F18; R11; C61
Keywords: Carbon emission; Carbon Taxes; Spatial Model

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1 Introduction

Ambitious carbon reduction targets, adopted by many countries,[1] may lead to decreased production efficiency and increased costs, as suggested by the literature.[2] However, the spatial allocation of these costs and reduction efforts remains an open question. For instance, should we concentrate the steel industry in one region that has comparative advantage and reduce carbon emissions elsewhere, or should we reduce emissions uniformly across regions? The answer to this type of questions is complicated because it depends on many regional characteristics such as productivity, amenity, industrial composition, and emission intensity. Figure 1 depicts regional variations in China's emission intensity. Regarding policy implications, it is unclear whether policy should target carbon reduction in a few high-intensity provinces or in many lower-intensity areas. Additionally, regions are interconnected via trade, migration, and input-output linkages. Reducing emission in one region and sector could affect production and emissions in all other regions and sectors. Creating a carbon reduction strategy that improves overall welfare must account for this heterogeneity and interdependence.

To understand the optimal spatial allocation of carbon emissions, we need a quantitative spatial model that incorporates carbon emissions and policies, as well as an efficient algorithm to solve for the optimal policies. In the literature, we still lack such a model,[3] and solving for high-dimensional optimal spatial policies remains computationally chal-

[1] For example, in 2022 Australia legislated its greenhouse gas emission reduction targets, aiming to reach emission levels of 43% below 2005 levels by 2030 and to reach net zero by 2050. In 2021, President Biden set the U.S. Greenhouse Gas Pollution Reduction Target aiming to reduce net greenhouse gas emissions by 50-52% from 2005 levels in 2030. In 2020, China proposed to reduce carbon emissions rapidly by 2045 and achieve carbon neutrality by 2060.

[2] Recent studies emphasizes energy as an input for production and carbon emission as a side output of energy usage. As a result, carbon reduction increases the cost of energy input and thereby the cost of production. See, for example, Larch and Wanner (2017).

[3] Farrokhi and Lashkaripour (2024) make progress in this direction by incorporating carbon emissions and policies into a quantitative trade model and characterizing optimal trade policies aiming to reduce global carbon emissions. Their framework follows the quantitative trade model with pollution emissions developed by Shapiro and Walker (2018). However, so far as we know, there is no quantitative framework designed to understand optimal carbon polices within country across regions.

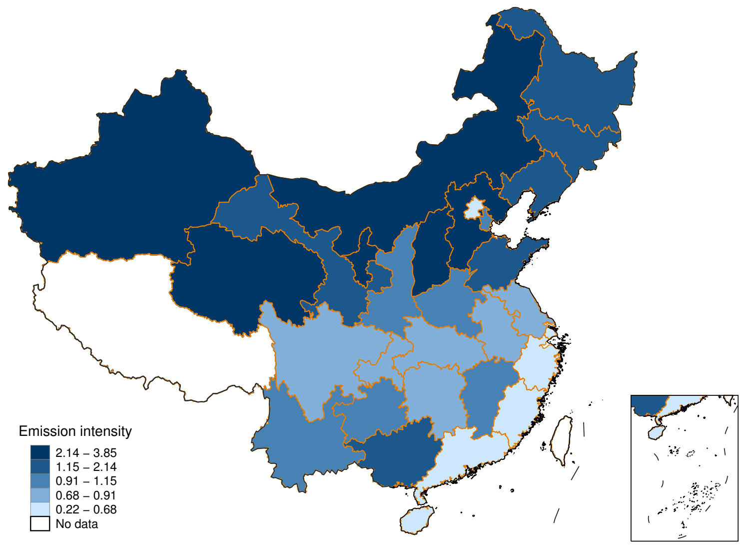

Figure 1: Carbon Emission Intensity in China (2017)

(Notes: Emission intensity = Emission share in China/GDP share in China. Data Source: Carbon Emission Account&Datasets (CEADs) and China's Statistical Yearbook.)

lenging.[4]

To fill these gaps, we develop a multi-sector quantitative spatial model with carbon emissions and carbon taxes. Building on the spatial model developed by Allen and Arkolakis (2014), we integrate carbon emissions and carbon taxes as specified in Shapiro and Walker (2018) and Farrokhi and Lashkaripour (2024). Our model simulates a spatial economy in which workers are free to move across regions, driven by differences in technology, amenities, and local agglomeration and congestion forces. Regions are connected through trade and input-output linkages, which facilitate the exchange of goods and services.

In our model, carbon emissions are considered a factor of production, and their price is determined by the government's carbon taxes. When a carbon tax is imposed on a specific sector in a particular region, it reduces carbon emissions in that sector and region, but increases its production costs. This tax also affects carbon emissions and production

[4]For instance, Lashkaripour and Lugovskyy (2023) highlight "the well-known limitations of numerical optimization routines when applied to nonlinear models with many free-moving variables".

costs in other sectors and regions through trade and input-output linkages, as goods and services are exchanged across the economy.

Leveraging the model above, we characterize carbon taxes that maximize aggregate real income, subject to an aggregate carbon emission constraint. To this end, we develop a novel emission allocative efficiency (EAE) measure, which quantifies the potential real income gains achievable through the reallocation of emissions. This measure assesses the trade-off between the benefits of reducing carbon emissions and the associated losses in production efficiency for any given set of spatial carbon taxes. We show that the optimal carbon tax for a particular sector in a given region is higher than the observed carbon tax in that sector and region if its EAE exceeds 1, and lower if its EAE is less than 1. This property of the EAE provides qualitative guidance for improving real income through emission reallocation without the need to solve for optimal spatial carbon taxes explicitly.

Our analysis of the EAE is twofold. First, we develop a sufficient statistics approach that links EAE with inter-regional trade, labor, and emission data. In a one-sector special case, we derived an analytical expression that correlates a region's EAE with its Katz-Bonacich centrality within the inter-regional trade network. This finding indicates that regions with greater influence should have lower carbon taxes, as reductions in these areas could significantly impact overall production efficiency. Furthermore, in our full model, we demonstrate that EAE can be calculated by solving a linear system using data on inter-regional trade, labor, and emissions. Overall, our sufficient statistics approach reveals that EAE has intuitive implications for designing optimal spatial carbon taxes and can be computed using readily accessible data.

Second, we propose an iterative algorithm for solving optimal spatial carbon taxes using the properties of EAE. This algorithm combines the linear system that solves for EAE with the nonlinear system that solves for equilibrium changes under exogenous changes (the "exact-hat" algebra). The algorithm utilizes the EAE to enhance the computation speed by informing the structure of the Jacobian matrix in the optimization process. This allows our algorithm to efficiently compute complex, high-dimensional, and continuous optimal policies within a general equilibrium setting. It can be applied in a wide range of quantitative trade and spatial models.

We apply our framework to quantify optimal carbon taxes for 30 provinces and 15 sectors in China using a calibrated model based on 2017 data on production, trade flows, input-output linkages, population, and carbon emissions. We calculate the EAE for the observed carbon taxes in the calibrated economy and then compute optimal carbon taxes using our iterative algorithm. We find that

(i) EAE is highly correlated with the ratio of optimal carbon taxes over observed carbon taxes. It suggests that EAE provides policy makers with qualitative guidance for reallocating emissions to boost real income, without calculating the precise optimal spatial carbon taxes.

(ii) Optimal spatial carbon taxes in our full model are negatively correlated with the Katz-Bonacich centrality within the observed inter-regional trade network. This finding implies that the readily computable Katz-Bonacich centrality metric can serve as a valuable indicator for guiding the design of spatially targeted emissions policies.

(iii) Implementing optimal spatial carbon taxes could increase Chinese welfare by 1.42% while keeping total emissions unchanged. This finding highlights the potential for substantial gains when policymakers take into account regional heterogeneity and interdependence in designing carbon reduction policies.

Related Literature–This paper relates to several strands of literature. First, it relates to quantitative explorations of policies on carbon emission. Shapiro (2021) and Garcia-Lembergman, Ramondo, Rodriguez-Clare, and Shapiro. (2024) quantify the impacts of carbon policies in the global economy. Farrokhi and Lashkaripour (2024) further consider the optimal design of carbon policies within a quantitative trade model. Our paper complements this strand of literature by focusing on the optimal design of carbon emission policies across different regions within a country, which, to the best of our knowledge, has not been extensively explored in previous studies.[5]

Second, we contribute to the characterization of optimal spatial policies. Fajgelbaum and Gaubert (2020) characterize optimal transfers in a generalized spatial framework. Henkel, Seidel, and Suedekum (2021) and Colas and Hutchinson (2021) investigate optimal taxes and fiscal transfers across different regions. Our paper enriches this body of literature by bridging the gap between theoretical characterizations and the implementation of crucial spatial policies in practice.

Third, our work relates to targeting interventions in networks. Galeotti, Golub, and Goyal (2020) provide generalized theoretical results for this problem. Liu (2019) examines the impacts of industrial policies in production networks, while Lashkaripour and Lugovskyy (2023) consider optimal industrial policies in trade networks. Liu and Ma (2024) investigate innovation subsidies in knowledge networks. This paper contributes to this strand of literature by deriving sufficient statistics that can be used to characterize optimal spatial carbon taxes through (i) linear approximation and (ii) a simple iterative algorithm. Our framework for linear approximation and the iterative algorithm can be applied to characterize a wide range of policies in networks, such as the combination of zoning and industrial policies, pollution reduction across different regions, and economic sanctions in trade and technology networks.

This paper is structured as follows. Section 2 introduces our model. Section 3 provides a characterization of optimal spatial carbon taxes. In Section 4, we calibrate our model and perform counterfactual analysis for optimal spatial emissions. Finally, we conclude in Section 5.

[5] One exception is Arkolakis and Walsh (2023). They investigate the optimal spatial allocation of electricity transmission networks and the corresponding consequences on the adoption of renewable energy. This paper departs from their work by considering generalized spatial policies on carbon emissions.

2 Spatial Model with Carbon Emission and Carbon Tax

2.1 Environment

Consider a spatial economy with N regions, denoted by (i, n, k), and J sectors, denote by (j, s). Total endowment of workers is $\bar{L}$. Workers are freely mobile across regions and sectors. The representative consumer in region i has a Cobb-Douglas preference over J sectors:

$$U_i = B_i L_i^{-\beta} \prod_{j=1}^{J} \left(C_i^j\right)^{\alpha_j}, \quad \sum_{j=1}^{N} \alpha_j = 1, \tag{1}$$

where C_i^j is the consumption of sector j in region i. $B_i L_i^{-\beta}$ represents amenity in region i, where B_i is the exogenous amenity shifter, L_i is the labor in region i, and $\beta \geq 0$ captures the congestion force over space.

Each sector j consists of a unit mass of varieties, aggregated by a CES function with the elasticity of substitution $\sigma_j \geq 0$. Following Shapiro and Walker (2018) and Farrokhi and Lashkaripour (2024), we regard carbon emission as a factor of production whose price is determined by carbon tax. This is a tractable way to incorporate carbon abatement costs and carbon policies into a general equilibrium framework. Specifically, we assume that each variety is produced by a firm using labor, carbon, and intermediates in a perfectly competitive market. The unit cost of variety ω of sector j produced in region i is given by

$$c_i^j(\omega) = \frac{c_i^j}{z_i^j(\omega)}, \quad c_i^j \equiv L_i^{-\psi_j} w_i^{\gamma_j^L} \prod_{s=1}^{J} (P_i^s)^{\gamma_{sj}} \left(t_i^j\right)^{\xi_j}, \quad \gamma_j^L + \sum_{s=1}^{J} \gamma_{sj} + \xi_j = 1, \tag{2}$$

where w_i is the wage in region i, P_i^s is the price index of sector s in region i, $t_i^j > 0$ is the tax rate on carbon emission in region i and sector j, and $z_i^j(\omega)$ is the productivity of variety ω. Notice that (i) $\psi_j \geq 0$ characterizes the sectoral agglomeration force,[6] and (ii) ξ_j is the share of carbon emission in producing good j which, as shown below, affects the emission

[6] This specification follows Adao, Arkolakis, and Esposito (2023) to allow productivities of different sectors respond differently to changes in local production scale.

intensity of sector j. We assume that region i has a share s_i in the carbon tax revenue. In our baseline specification, we assume that $s_i = \frac{w_i L_i}{\sum_{k=1}^{N} w_k L_k}$.

The exogenous productivity $z_i^j(\omega)$ is drawn independently from a Frechet distribution with level parameter A_i^j and shape parameter $\theta_j \geq \sigma_j$. Exporting good j from region i to n incurs an iceberg trade cost τ_{in}^j.

2.2 Equilibrium

We proceed by defining the equilibrium in our model. Let X_i^j be the total expenditure in region i on good j and X_{in}^j be the value of trade of good j from region i to n. Then

$$\lambda_{in}^j \equiv \frac{X_{in}^j}{X_n^j} = \frac{A_i^j \left(\tau_{in}^j c_i^j\right)^{-\theta_j}}{\sum_{k=1}^{N} A_k^j \left(\tau_{kn}^j c_k^j\right)^{-\theta_j}}. \tag{3}$$

The price indices can be expressed as

$$P_n^j = \left[\sum_{k=1}^{N} A_k^j \left(\tau_{kn}^j c_k^j\right)^{-\theta_j}\right]^{-\frac{1}{\theta_j}}, \quad P_n = \prod_{j=1}^{J} \left(P_n^j\right)^{\alpha_j}. \tag{4}$$

The wage satisfies

$$w_i L_i = \sum_{j=1}^{J} \gamma_j^L \sum_{n=1}^{N} \lambda_{in}^j X_n^j. \tag{5}$$

Final income in region i is the sum of wage income and carbon tax revenue:

$$Y_i = w_i L_i + s_i R, \quad R \equiv \sum_{k=1}^{N} \sum_{j=1}^{J} R_k^j, \quad R_k^j \equiv \xi_j \sum_{n=1}^{N} \lambda_{kn}^j X_n^j, \quad s_i = \frac{w_i L_i}{\sum_{k=1}^{N} w_k L_k}. \tag{6}$$

X_i^j is the sum of final consumption and intermediate usage:

$$X_i^j = \alpha_j Y_i + \sum_{s=1}^{J} \gamma_{js} \sum_{n=1}^{N} \lambda_{in}^s X_n^s. \tag{7}$$

Welfare equalization implies that

$$B_i L_i^{-\beta} \frac{Y_i/L_i}{P_i} = W. \tag{8}$$

Since $\bar{L} = \sum_{i=1}^{N} L_i$, we have

$$\frac{L_i}{\bar{L}} = \frac{\left(B_i \frac{Y_i/L_i}{P_i}\right)^{\frac{1}{\beta}}}{\sum_{k=1}^{N} \left(B_k \frac{Y_k/L_k}{P_k}\right)^{\frac{1}{\beta}}}, \tag{9}$$

and the aggregate welfare can be measured by the weighted average of regional real income:

$$W = \frac{1}{\bar{L}^{\beta}} \left[\sum_{k=1}^{N} \left(B_k \frac{Y_k/L_k}{P_k}\right)^{\frac{1}{\beta}}\right]^{\beta}. \tag{10}$$

Finally, the aggregate emission is given by

$$E = \sum_{j=1}^{J} \sum_{i=1}^{N} E_i^j, \quad E_i^j \equiv \frac{\xi_j}{t_i^j} \sum_{n=1}^{N} \lambda_{in}^j X_n^j. \tag{11}$$

Definition Given parameters $\left(\psi_j, \beta, \theta_j, \alpha_j, \gamma_j^L, \gamma_{sj}, \xi_j; A_i^j, B_i, \tau_{in}^j; \bar{L}; t_i^j\right)$, the equilibrium consists of $\left(w_i, L_i, P_n^j, X_n^j\right)$ such that (i) (w_i) is given by labor market clearing in Equation (5); (ii) (L_i) is given by the labor allocation in Equation (9); (iii) $\left(P_n^j\right)$ is given by the price index in Equation (4); (iv) $\left(X_n^j\right)$ is given by goods market clearing in Equation (7).

Following Dekle, Eaton, and Kortum (2008), we can express our equilibrium system in relative changes. For any variable $Z > 0$, we denote Z' as its level after changes and $\hat{Z} \equiv \frac{Z'}{Z}$. Let $\chi_{in}^j \equiv \frac{\gamma_j^L \lambda_{in}^j X_n^j}{w_i L_i}$ be the export share. Let $\delta_{kn}^j \equiv \frac{1}{R} \xi_j \lambda_{kn}^j X_n^j$ be the carbon tax revenue share. Let $\nu_{ij}^{Y} \equiv \frac{\alpha_j Y_i}{X_i^j}$ be final expenditure share. Let $\nu_{in}^{js} \equiv \frac{\gamma_{js}\lambda_{in}^{s} X_n^s}{X_i^j}$ be intermediate expenditure share.

Then given exogenous changes $\left(\hat{t}_i^j\right)$, we can derive $\left(\hat{w}_i, \hat{L}_i, \hat{P}_n^j, \hat{X}_n^j\right)$ by solving the following non-linear system:

$$\begin{aligned}
&\hat{w}_i \hat{L}_i = \sum_{j=1}^{J}\sum_{n=1}^{N} \chi_{in}^{j} \hat{\lambda}_{in}^{j} \hat{X}_n^j, \quad \hat{\lambda}_{in}^{j} = \left(\hat{c}_i^j\right)^{-\theta_j}\left(\hat{P}_n^j\right)^{\theta_j}, \quad \hat{c}_i^j = \hat{L}_i^{-\psi_j} \hat{w}_i^{\gamma_j^L} \left(\hat{t}_i^j\right)^{\xi_j} \prod_{s=1}^{J} \left(\hat{P}_i^s\right)^{\gamma_{sj}} \\
&\left(\hat{P}_n^j\right)^{-\theta_j} = \sum_{i=1}^{N} \lambda_{in}^{j} \left(\hat{c}_i^j\right)^{-\theta_j}, \quad \hat{P}_n = \prod_{j=1}^{J} \left(\hat{P}_n^j\right)^{\alpha_j} \\
&\hat{X}_i^j = \nu_{ij}^{Y} \hat{Y}_i + \sum_{s=1}^{J}\sum_{n=1}^{N} \nu_{in}^{js} \hat{\lambda}_{in}^{s} \hat{X}_n^s, \quad \hat{Y}_i = \hat{w}_i \hat{L}_i \left[\frac{1}{1+R} + \frac{R}{1+R} \sum_{k=1}^{N}\sum_{j=1}^{J}\sum_{n=1}^{N} \delta_{kn}^{j} \hat{\lambda}_{kn}^{j} \hat{X}_n^j\right], \qquad (12) \\
&\hat{L}_i = \frac{\left(\frac{\hat{w}_i}{\hat{P}_i}\right)^{\frac{1}{\beta}}}{\sum_{k=1}^{N} \iota_k \left(\frac{\hat{w}_k}{\hat{P}_k}\right)^{\frac{1}{\beta}}}, \quad \iota_i \equiv \frac{L_i}{\bar{L}}.
\end{aligned}$$

3 Optimal Spatial Emissions

3.1 Optimization Problem and Emission Allocative Efficiency (EAE)

The central government decides $\left(t_i^j\right)$ to maximize the aggregate welfare subject to an aggregate emission constraint:

$$\begin{aligned}
&\max_{\left(t_i^j; w_i, L_i, P_n^j, X_n^j\right)} W \equiv \frac{1}{\bar{L}^{\beta}} \left[\sum_{k=1}^{N} \left(B_k \frac{Y_k / L_k}{P_k}\right)^{\frac{1}{\beta}}\right]^{\beta} \\
&\text{s.t. } \sum_{j=1}^{J}\sum_{i=1}^{N} \frac{\xi_j}{t_i^j} \sum_{n=1}^{N} \lambda_{in}^{j} X_n^j \leq \bar{E}, \qquad (13) \\
&\left(w_i, L_i, P_n^j, X_n^j\right) \text{ satisfy Equation (4), (5),(9), and (7)}
\end{aligned}$$

Given the Cobb-Douglas utility function and production function, it is straightforward to show that the solution to Problem (13), denoted as $\mathbf{t}^* \equiv \left(t_i^{j*}\right)$, exists.

To characterize the optimal carbon taxes $\mathbf{t}^*$, we develop an emission allocative efficiency (EAE) measure: for any carbon tax profile $\mathbf{t} \equiv \left(t_i^j\right)$, its corresponding EAE is defined as

$$M_i^j(\mathbf{t}) \equiv \frac{\mu}{W} \left[\underbrace{E_i^j + \sum_{s=1}^{J} \sum_{k=1}^{N} \left(-\frac{\partial \log R_k^s}{\partial \log t_i^j} \right) E_k^s}_{\text{Effect of } t_i^j \text{ on carbon emissions}} \right] \left(\underbrace{-\frac{\partial \log W}{\partial \log t_i^j}}_{\text{Effect of } t_i^j \text{ on real income}} \right)^{-1}, \tag{14}$$

where μ is the Lagrange multiplier of the aggregate carbon emission constraint in Equation (13).

By construction, $M_i^j(\mathbf{t})$ increases with the effect of t_i^j on aggregate carbon emission and decrease with (the absolute value of) the effect of t_i^j on aggregate real income W. As a result, $M_i^j(\mathbf{t})$ summarizes the key trade-off in determining carbon taxes: the increase in t_i^j would lower carbon emissions but also lower real income by raising production costs. In the following lemma, we will argue that $M_i^j(\mathbf{t})$ measures the extent to which emission reallocation would increase the aggregate real income.

Proposition 1 (Emission Allocative Efficiency) *Let $\mathbf{t}^* \equiv \left(t_i^{j*}\right)$ be the solution of Problem* (13). *Then there exists $\delta > 0$ such that for any $\mathbf{t}$ satisfying $\sum_{i,j} \left[t_i^j - t_i^{j*}\right]^2 \leq \delta$, $M_i^j(\mathbf{t})$ defined by Equation* (14) *has the following properties:*

1. $M_i^j(\mathbf{t}^*) = 1$ *for all* (i, j).
2. *If* $M_i^j(\mathbf{t}) > 1$, *then* $t_i^{j*} > t_i^j$.
3. *If* $M_i^j(\mathbf{t}) < 1$, *then* $t_i^{j*} < t_i^j$.

Proposition 1 suggests that t_i^j should increase if the benefit from carbon reduction exceeds the loss from lowering real income. Moreover, Proposition 1 implies that t_i^{j*} is higher in the region-sector pair where (i) the carbon tax can substantially reduce carbon emission, or (ii) the carbon tax has small negative effects on the aggregate welfare.

Though Lemma 1 holds only if $\mathbf{t}$ is close to $\mathbf{t}^*$, our counterfactual analysis in Section 4.2 will show that it holds numerically in most of the (i, j)-pairs for the observed $\mathbf{t}$ in our quantification practice. Consequently, $M_i^j(\mathbf{t})$ could offer a qualitative guidance for spatial carbon policies: carbon taxes should be higher in region-sector pairs with higher $M_i^j(\mathbf{t})$ and lower in those with lower EAE.

Although EAE is useful in characterizing optimal spatial carbon taxes, it is a general equilibrium outcomes determined by rich heterogeneity and interdependence across regions. To make EAE useful in quantification, we first take a sufficient statistics approach by connecting EAE with data observations. Then, based on these sufficient statistics, we develop an iterative algorithm to solve for the optimal spatial carbon taxes.

3.2 Sufficient Statistics

In this subsection, we express $M_i^j(\mathbf{t})$ by model parameters $\left(\psi_j, \beta, \theta_j, \alpha_j, \gamma_j^L, \gamma_{sj}, \xi_j\right)$ and data on trade, labor, and emissions, $\left(X_{in}^j, L_i, E_i^j\right)$. We first derive analytical expressions of EAE in the one-sector special case of our model. We then derive a linear system that can be used to solve EAE in our full model.

One-sector special case. Consider there is only one sector, *i.e.* $J = 1$ and there is no roundabout production, *i.e.* $\gamma_{js} = 0$ for all (j, s). We omit subscript/superscript j for all variables in this case. The EAE in this case can be expressed as

$$M_i(\mathbf{t}) = \frac{\mu\xi}{W}\left[E_i + \sum_{k=1}^{N}\left(-\frac{\partial \log R_k}{\partial \log t_i}\right)E_k\right]\left(-\frac{\partial \log W}{\partial \log t_i}\right)^{-1}. \tag{15}$$

We define several parameters and matrices. Let $\breve{\theta} = \frac{\theta}{1+\theta(2-\xi)}$, $\delta_1 = 1 + [1 + \theta(1-\xi)]\,\beta - \theta\psi$, and $\delta_2 = (1-\xi) - \theta(1-\xi)\beta + (1+\theta)\psi$. Let $\boldsymbol{\iota} \equiv [\iota_i]$ be a column vector, $\mathbf{e}$ be a column vector with all ones, and $\mathbf{I}$ be the identity matrix. Let $\chi_{in} \equiv \frac{X_{in}}{\sum_{k=1}^{N} X_{ik}}$, and $\boldsymbol{\chi} = [\chi_{in}]$ with i denoting the rows and n denoting the columns.

Proposition 2 *The welfare effects of carbon taxes:*

$$\left[\frac{\partial \log W}{\partial \log t_1}, \ldots, \frac{\partial \log W}{\partial \log t_N}\right] = -\frac{\boldsymbol{\iota}' \left[\mathbf{I} - \frac{\delta_2}{\delta_1}\boldsymbol{\chi}\right]^{-1} \left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\boldsymbol{\chi}\right]}{\frac{\theta}{\breve{\theta}\delta_1}\boldsymbol{\iota}' \left[\mathbf{I} - \frac{\delta_2}{\delta_1}\boldsymbol{\chi}\right]^{-1} \mathbf{e}} \tag{16}$$

The effects of carbon taxes on carbon tax revenues

$$\left[\frac{\partial \log R_k}{\partial \log t_i}\right] = \left[1 + \breve{\theta}\left(\beta + \psi - \frac{1}{\theta}\right)\right]\left[\frac{\partial \log L_k}{\partial \log t_i}\right] - \breve{\theta}\xi\mathbf{I}, \tag{17}$$

where $\left[\frac{\partial \log L_k}{\partial \log t_i}\right] = -\left[\mathbf{I} - \frac{1}{\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\boldsymbol{\chi}\right]^{-1}\mathbf{e}} \left[\mathbf{I} - \frac{\delta_2}{\delta_1}\boldsymbol{\chi}\right]^{-1} \mathbf{e}\boldsymbol{\iota}'\right] \left[\mathbf{I} - \frac{\delta_2}{\delta_1}\boldsymbol{\chi}\right]^{-1} \left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\boldsymbol{\chi}\right]$.

The detailed proof to Proposition 2 is presented in Appendix A.2. Notably, matrix $\boldsymbol{\chi}$ represents inter-provincial trade networks and thereby summarizes rich heterogeneity and interdependence across regions. The vector $\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\boldsymbol{\chi}\right]^{-1}$ is the Katz-Bonacich Centrality (KC) that reflects the influence of each region in inter-provincial trade networks.

EAE in our full model. Characterizing EAE in our full model is challenging since there are multiple sectors with input-output linkages. Instead of analytical expressions, we express $\left(\frac{\partial \log R_k^s}{\partial \log t_i^j}\right)$ and $\left(\frac{\partial \log W}{\partial \log t_i^j}\right)$ in our full model as the solution to a linear recursive system.

Without loss of generality, we normalize $\sum_{i=1}^{N} w_i L_i = 1$. For any variables $Z_i > 0$, we denote $\tilde{Z}_i \equiv d\log Z_i$. Then $\left(\tilde{w}_i, \tilde{L}_i, \tilde{P}_i^j, \tilde{X}_i^j\right)$ can be computed by solving:

$$\tilde{w}_i + \tilde{L}_i = \sum_{j=1}^{J}\sum_{n=1}^{N}\chi_{in}^j\left(\tilde{\lambda}_{in}^j + \tilde{X}_n^j\right), \quad \tilde{\lambda}_{in}^j = -\theta_j\tilde{c}_i^j + \theta_j\tilde{P}_n^j, \quad \tilde{c}_i^j = -\psi_j\tilde{L}_i + \gamma_j^L\tilde{w}_i + \xi_j\tilde{t}_i^j + \sum_{s=1}^{J}\gamma_{sj}\tilde{P}_i^s$$
$$\tilde{X}_i^j = \nu_{ij}^Y\tilde{Y}_i + \sum_{s=1}^{J}\sum_{n=1}^{N}\nu_{in}^{js}\left(\tilde{\lambda}_{in}^s + \tilde{X}_n^s\right), \quad \tilde{Y}_i = \left(\tilde{w}_i + \tilde{L}_i\right) + \frac{R}{1+R}\sum_{k=1}^{N}\sum_{j=1}^{J}\sum_{n=1}^{N}\delta_{kn}^j\left(\tilde{\lambda}_{kn}^j + \tilde{X}_n^j\right)$$
$$\tilde{P}_n^j = \sum_{i=1}^{N}\lambda_{in}^j\tilde{c}_i^j, \quad \tilde{P}_n = \sum_{j=1}^{J}\alpha_j\tilde{P}_n^j, \quad \tilde{L}_i = \frac{1}{\beta}\left(\tilde{w}_i - \tilde{P}_i\right) - \frac{1}{\beta}\sum_{k=1}^{N}\iota_k\left(\tilde{w}_k - \tilde{P}_k\right) \tag{18}$$

The welfare effects of carbon taxes can then be derived by:

$$\tilde{W} = \sum_{k=1}^{N}\iota_k\left(\tilde{Y}_k - \tilde{L}_k - \tilde{P}_k\right). \tag{19}$$

The impacts of carbon taxes on carbon tax revenues can be expressed as

$$\tilde{R}_i^j = \sum_{n=1}^{N}\frac{\xi_j\lambda_{in}^j X_n^j}{R_i^j}\left(\tilde{\lambda}_{in}^j + \tilde{X}_n^j\right). \tag{20}$$

Therefore, for any spatial economy where we can observe $\left(\psi_j, \beta, \theta_j, \alpha_j, \gamma_j^L, \gamma_{sj}, \xi_j\right)$ and $\left(X_{in}^j, L_i, E_i^j\right)$, $M_i^j(\mathbf{t})$ defined by Equation (14) can be derived by solving Equation (18).

3.3 An Iterative Algorithm to Solve for t_i^{j*}

In this subsection, we utilize the property of EAE shown in Proposition 1 to develop an iterative algorithm solving for $\left(t_i^{j*}\right)$.

Algorithm 3 $\left(t_i^{j*}\right)$ *that solves the problem in Equation (13) can be calculated as follows:*

1. *Guess* $\left(t_i^{j*}\right) \in \mathbb{R}_{++}^{N\times J}$.
2. *Solve for* $\left(X_{in}^j, \iota_i\right)$ *under* $\left(t_i^{j*}\right)$ *by "exact-hat algebra" in Equation (12).*

3. *Solve the linear system* (18).

4. *Calculate* $M_i^j(\mathbf{t}^*)$ *using Equation* (19), (20), *and* (14).

5. *Update* t_i^{j*} *by* $t_i^{j*}M_i^j(\mathbf{t}^*)$.

6. *Repeat Step 1-5 until* $t_i^{j*} = t_i^{j*}M_i^j(\mathbf{t}^*)$ *for all* (i, j).

7. *Adjust the level of* $\left(t_i^{j*}\right)$ *to bind the aggregate emission constraint in Equation* (13).

4 Quantification

4.1 Data and Calibration

Our quantitative analysis requires values on $\left(\psi_j, \beta, \theta_j, \alpha_j, \gamma_j^L, \gamma_{sj}; \xi_j, t_i^j; X_{in}^j\right)$. We consider $N = 30$ Chinese provinces and $J = 15$ sectors in 2017. We calibrate (θ_j, ψ_j) from Lashkaripour and Lugovskyy (2023). Notably, we rescale the scale elasticities so that its average is equal to 0.05, consistent with the estimate in Adao et al. (2023). We report the calibrated values of (θ_j, ψ_j) in the first two columns of Table 1.

We calibrate $\beta = \frac{2}{3}$ from Tombe and Zhu (2019). We calibrate $\left(\alpha_j, \gamma_j^L, \gamma_{sj}\right)$ using Chinese aggregate input-output table for 2017. We obtain $\left(X_{in}^j\right)$ directly from Chinese inter-provincial input-output table for 2017.

We calibrate $\left(\xi_j, t_i^j\right)$ combining Chinese inter-provincial input-output table and Carbon Emission Account&Datasets (CEADs). In particular, we have

$$\frac{\xi_j}{t_i^j} = \frac{E_i^j}{\sum_{n=1}^N X_{in}^j}. \tag{21}$$

We normalize $\frac{1}{N}\sum_{i=1}^N t_i^j = 1$ for all j. We then get ξ_j and t_i^j separately. To this end, we attribute all sectoral variations in emission intensities to (ξ_j). Notably, we adjust the unit

Table 1: Calibration of $\left(\theta_j, \psi_j, \xi_j\right)$

Sector	Description	θ_j	ψ_j	ξ_j
1	Agriculture&Mining	6.227	0.0254	0.0203
2	Food	2.303	0.0697	0.0064
3	Textiles, Leather&Footwear	3.359	0.0397	0.0083
4	Wood	3.896	0.0406	0.0032
5	Paper	2.646	0.0567	0.0153
6	Petroleum	1.200	0.2163	0.0368
7	Chemicals	3.966	0.0411	0.0262
8	Rubber&Plastic	5.157	0.0248	0.0041
9	Minerals	5.283	0.0296	0.0883
10	Basic&Fabricated Metals	3.004	0.0371	0.0635
11	Machinery	7.75	0.0213	0.0077
12	Electrical&Optical Equipment	1.235	0.0979	0.0034
13	Transport Equipment	2.805	0.0229	0.0041
14	N.E.C.&Recycling	6.169	0.0270	0.0054
15	Services	10	0.0000	0.0070
	Simple Average	4.33	0.05	0.02

of E_i^j so that $\frac{1}{J}\sum_{j=1}^{J}\xi_j = 0.02$, consistent with the estimate in Shapiro and Walker (2018). We report the calibrated values of $\left(\xi_j\right)$ in the last column of Table 1.

4.2 EAE and Optimal Spatial Carbon Taxes

In this subsection, we characterize $\left(t_i^{j*}\right)$ in our calibrated economy. First, we calculate EAE $M_i^j(\mathbf{t})$ for the observed $\left(t_i^j\right)$ by solving the linear system in Equation (18). Notably, to solve for the partial derivatives $\frac{\partial \log W}{\partial \log t_i^j}$ and $\frac{\partial \log R_k^s}{\partial \log t_i^j}$, we need to solve Equation (18) for $N \times J$ times. It takes about 3 minutes in a personal computer to calculate $M_i^j(\mathbf{t})$ under $N = 30$ and $J = 15$.

Figure 2 depicts $M_i^j(\mathbf{t})$ for the observed $\left(t_i^j\right)$ in our calibrated economy. It suggests that carbon taxes should increase in few provinces such as Shanxi, Xinjiang, Inner Mongolia, and Liaoning. These provinces are with high emission intensities and are relatively peripheral within inter-regional trade networks. As a result, raising t_i^j in these provinces could lead to large gains from carbon reduction and small losses in production efficiency.

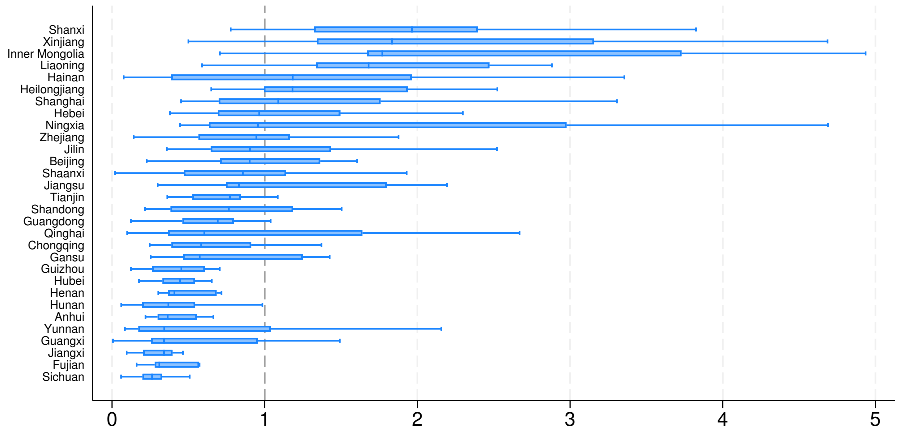

Figure 2: Emission Allocative Efficiency $M_i^j(\mathbf{t})$ in 2017 Chinese Economy

(Notes: The bar in the middle of the box indicates the median of $M_i^j(\mathbf{t})$ for province i. Two edges of the box represent, respectively, 25 and 75 percentiles. Notice that $M_i^j(\mathbf{t}) > 1$ suggests that $t_i^{j*} > t_i^j$.)

We then compute $\left(t_i^{j*}\right)$ using Algorithm 3. The algorithm takes about 3 hours to converge under $N = 30$ and $J = 15$. Comparing with the algorithm used in Ossa (2014), our algorithm is efficient in solving for the high-dimensional optimal policies.

Figure 3 depicts $\left(t_i^{j*}\right)$ in 2017 Chinese economy. It suggests that within each sector $\left(t_i^{j*}\right)$ vary substantially across provinces. The significant regional disparities in $\left(t_i^{j*}\right)$ underscore the critical role of spatial dimensions in designing welfare-enhancing carbon tax policies. Moreover, optimal carbon taxes would increase Chinese real income by 1.42% compared to the calibrated economy, while keeping aggregate carbon emissions unchanged.

Proposition 2 has revealed an analytical relationship between $\left(t_i^{j*}\right)$ and Katz-Bonacich centrality within inter-regional trade networks in a one-sector special case. Figure 4 depicts this relationship in our full model, suggesting that t_i^{j*} is significantly negatively correlated with Katz-Bonacich centrality within inter-provincial trade networks. Intuitively, the region-sector pair with higher Katz-Bonacich centrality has larger influence within inter-regional trade networks. This result demonstrates the informativeness of

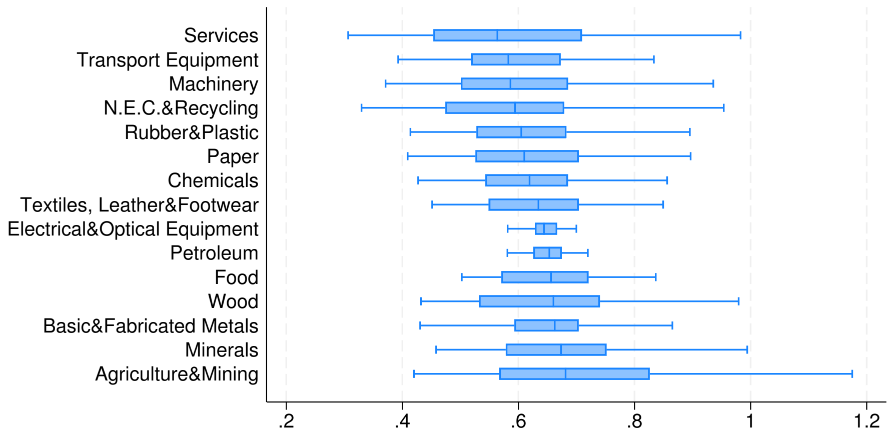

Figure 3: Optimal Spatial Carbon Taxes in 2017 Chinese Economy

(Notes: The bar in the middle of the box indicates the median of $\left(t_i^{j*}\right)$ for sector j. Two edges of the box represent, respectively, 25 and 75 percentiles.)

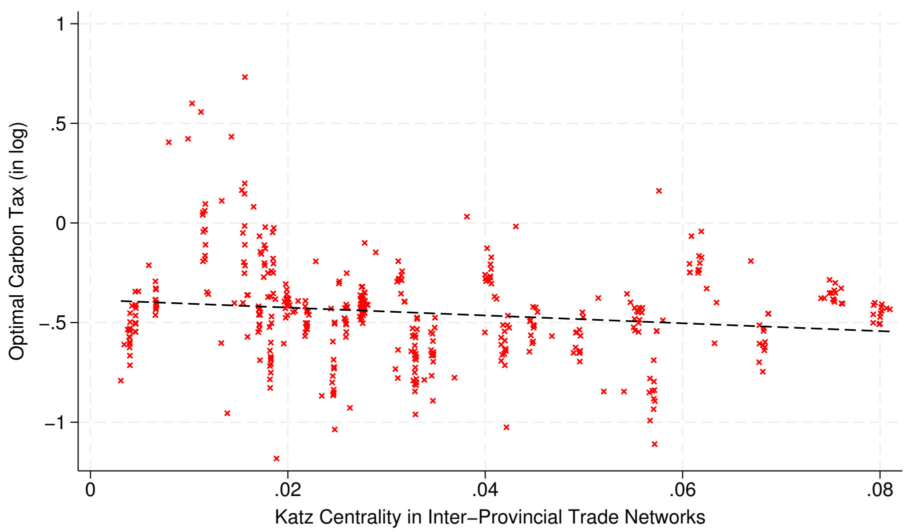

Figure 4: Optimal Spatial Carbon Taxes and Katz-Bonacich Centrality within Inter-Provincial Trade Networks

(Notes: $\left[KC_1^j, \ldots, KC_N^j\right] \equiv \boldsymbol{\iota}' \left[\mathbf{I} - \frac{\delta_{2j}}{\delta_{1j}} \chi_j\right]^{-1}$. The dash line shows the linear fit. Regressing $\log\left(t_i^{j*}\right)$ on KC_i^j results in the slope coefficient -1.96 with s.e. 0.56.)

readily computable Katz-Bonacich centrality about welfare-enhancing reallocation of carbon emissions.

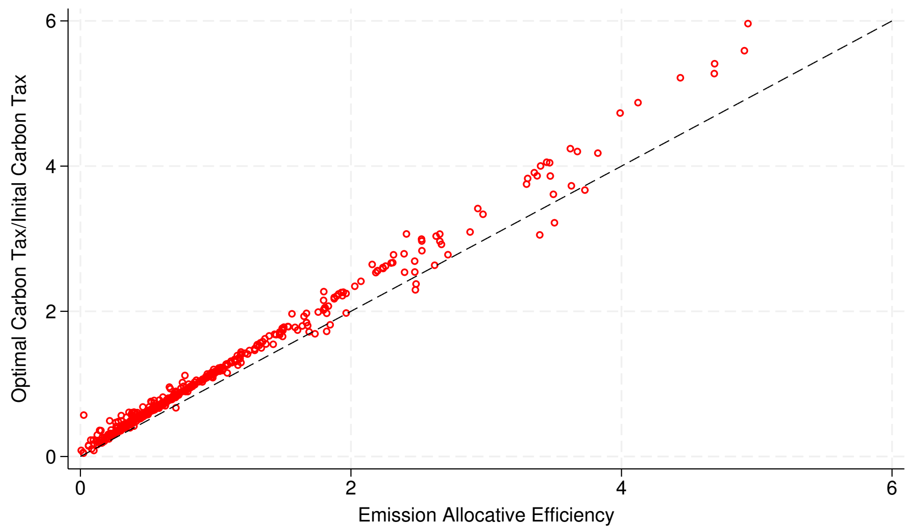

Figure 5: Optimal Spatial Carbon Taxes and Emission Allocative Efficiency

(Notes: The dash line is the 45-degree line.)

Finally, we investigate to what extent our EAE, $M_i^j(\mathbf{t})$, for the observed $\left(t_i^j\right)$ can predict $\left(t_i^{j*}/t_i^j\right)$. Figure 5 suggests that $M_i^j(\mathbf{t})$ is highly significantly correlated with $\left(t_i^{j*}/t_i^j\right)$. Therefore, using our EAE $M_i^j(\mathbf{t})$ offers a highly efficient approach to guide spatial carbon policies, taking just a fraction (1/60th) of the time required for computing optimal spatial carbon taxes.

5 Conclusion

In this paper, we develop a multi-sector quantitative spatial model with carbon emissions and carbon taxes to quantify optimal spatial carbon policies. We propose a novel emission allocative efficiency (EAE) measure that quantifies the extent to which emission reallocation could increase real income. We develop a sufficient statistics approach that connects EAE with readily computable Katz-Bonacich centrality within inter-regional trade networks and, based on the properties of EAE, design an iterative algorithm to compute high-dimensional optimal spatial carbon taxes. Our quantitative analysis of the Chinese economy highlights the importance of inter-regional connections for optimal allocation of carbon emissions.

Our framework has broad policy applications. First, it can calculate optimal spatial carbon policies for major economies including the U.S. and EU. Second, it can be used to compute optimal policies in other networks such as industrial policies in trade and production networks and innovation policies in knowledge networks. Notably, we do not specify how the optimal spatial carbon taxes could be practically implemented. This implementation question remains open for future research.

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A Theory

A.1 Emission Allocative Efficiency

The Lagrange function of Problem (13) is defined as

$$\mathcal{L}(\mathbf{t};\mu) \equiv \frac{1}{\bar{L}^{\beta}} \left[\sum_{k=1}^{N} \left(B_k \frac{Y_k/L_k}{P_k} \right)^{\frac{1}{\beta}} \right]^{\beta} + \mu \left[\bar{E} - \sum_{j=1}^{J} \sum_{i=1}^{N} \frac{\xi_j}{t_i^j} \sum_{n=1}^{N} \lambda_{in}^j X_n^j \right]. \tag{A1}$$

The first-order conditions indicate that the optimal carbon taxes $\left(t_i^{j*}\right)$ satisfy:

$$\frac{\partial \mathcal{L}(\mathbf{t}^*;\mu^*)}{\partial t_i^j} = 0 \Rightarrow \frac{\partial W}{\partial t_i^j} = \mu^* \left[-\frac{R_i^{j*}}{\left(t_i^{j*}\right)^2} + \sum_{s=1}^{J} \sum_{k=1}^{N} \frac{\partial R_k^s}{\partial t_i^j} \frac{1}{t_k^{s*}} \right], \tag{A2}$$

and

$$\sum_{j=1}^{J} \sum_{i=1}^{N} E_i^{j*} = \bar{E}. \tag{A3}$$

Proof to Proposition 1

Equation (A2) can be expressed as:

$$1 = \frac{\mu^*}{W^*} \left[E_i^{j*} + \sum_{s=1}^{J} \sum_{k=1}^{N} \left(-\frac{\partial \log R_k^s}{\partial \log t_i^j} \right) E_k^{s*} \right] \left(-\frac{\partial \log W}{\partial \log t_i^j} \right)^{-1}. \tag{A4}$$

The RHS of Equation (A4) is, by construction, $M_i^j(\mathbf{t}^*)$. Therefore, we have $M_i^j(\mathbf{t}^*) = 1$.

Suppose that $\mathbf{t} = \mathbf{t}^*$ except for t_i^j. Notice that $\mathcal{L}(\mathbf{t};\mu)$ is continuously differentiable w.r.t. $\mathbf{t}$. Then there exists $\delta > 0$ such that for any $t_i^j \in \left(t_i^{j*} - \delta, t_i^{j*} + \delta\right)$ we have

$$\frac{\partial \mathcal{L}(\mathbf{t};\mu)}{\partial t_i^j} > 0 \Rightarrow t_i^j < t_i^{j*} \quad \text{and} \quad \frac{\partial \mathcal{L}(\mathbf{t};\mu)}{\partial t_i^j} < 0 \Rightarrow t_i^j > t_i^{j*}. \tag{A5}$$

Equivalently, for any $t_i^j \in \left(t_i^{j*} - \delta, t_i^{j*} + \delta\right)$, we have if $M_i^j(\mathbf{t}) > 1$ then $t_i^{j*} > t_i^j$ and if $M_i^j(\mathbf{t}) < 1$ then $t_i^{j*} < t_i^j$.

Again, since $\mathcal{L}(\mathbf{t}; \mu)$ is continuously differentiable w.r.t. $\mathbf{t}$, there exists $\delta > 0$ such that for any $\mathbf{t}$ satisfying $\sum_{i,j} \left[t_i^j - t_i^{j*}\right]^2 \leq \delta$, we have if $M_i^j(\mathbf{t}) > 1$ then $t_i^{j*} > t_i^j$ and if $M_i^j(\mathbf{t}) < 1$ then $t_i^{j*} < t_i^j$.

Q.E.D.

A.2 One-Sector Case

In the one-sector case, the central government solves for the following spatial emission problem:

$$
\begin{aligned}
&\max_{(t_i; w_i, L_i, X_i)} W \equiv \frac{1}{\bar{L}^\beta} \left[\sum_{k=1}^N \left(B_k \frac{X_k/L_k}{P_k}\right)^{\frac{1}{\beta}}\right]^\beta \\
\text{s.t. } &\sum_{i=1}^N \frac{\xi}{t_i} \sum_{n=1}^N \lambda_{in} X_n \leq \bar{E}, \\
&w_i L_i = \gamma^L \sum_{n=1}^N \lambda_{in} X_n \\
&X_i = w_i L_i + s_i R, \quad R \equiv \xi \sum_{k=1}^N \sum_{n=1}^N \lambda_{kn} X_n, \quad s_i = \frac{w_i L_i}{\sum_{k=1}^N w_k L_k} \qquad \text{(A6)} \\
&P_n = \left[\sum_{k=1}^N A_k L_k^{\theta\psi} \left(\tau_{kn} w_k^{\gamma^L} t_k^{\xi}\right)^{-\theta}\right]^{-\frac{1}{\theta}} \\
&\frac{L_i}{\bar{L}} = \frac{\left(B_i \frac{X_i/L_i}{P_i}\right)^{\frac{1}{\beta}}}{\sum_{k=1}^N \left(B_k \frac{X_k/L_k}{P_k}\right)^{\frac{1}{\beta}}}
\end{aligned}
$$

Lagrange:

$$
\mathcal{L} = W + \mu \left(\bar{E} - \sum_{k=1}^N \frac{\xi X_k}{t_k}\right). \tag{A7}
$$

F.O.C.

$$
\frac{\partial W}{\partial t_i} + \mu \left(\frac{\xi X_i}{t_i^2} - \sum_{k=1}^N \frac{\xi}{t_k} \frac{\partial X_k}{\partial t_i}\right) = 0. \tag{A8}
$$

Then

$$-\frac{\partial \log W}{\partial \log t_i} = \frac{\mu}{W^*}\left(\frac{\xi X_i^*}{t_i^*} - \sum_{k=1}^{N}\frac{\xi X_k^*}{t_k^*}\frac{\partial \log X_k}{\partial \log t_i}\right). \tag{A9}$$

Then

$$t_i^* = \frac{\mu\xi}{W^*}\left[X_i^* + \sum_{k=1}^{N}\frac{t_i^*}{t_k^*}\left(-\frac{\partial \log X_k}{\partial \log t_i}\right)X_k^*\right]\left(-\frac{\partial \log W}{\partial \log t_i}\right)^{-1}. \tag{A10}$$

Proof to Proposition 2

Notice that the aggregate consumption expenditure is equal to the aggregate production value, *i.e.* $\sum_{i=1}^{N} X_i = \sum_{i=1}^{N}\sum_{n=1}^{N}\lambda_{in}X_n$. Also $\gamma^L + \xi = 1$. Therefore, we have $R = \frac{\xi}{1-\xi}\sum_{i=1}^{N} w_i L_i$ and $X_i = \frac{1}{1-\xi}w_i L_i$. Then the equilibrium system can be expressed in terms of $(w_i, L_i, P_i; W)$:

$$\begin{aligned}
w_i L_i &= \sum_{n=1}^{N} A_i L_i^{\theta\psi}\left(\tau_{in} w_i^{1-\xi} t_i^{\xi}\right)^{-\theta} P_n^{\theta} w_n L_n \\
P_i^{-\theta} &= \sum_{n=1}^{N} A_n L_n^{\theta\psi}\left(\tau_{ni} w_n^{1-\xi} t_n^{\xi}\right)^{-\theta} \\
L_i &= \left(\frac{1}{1-\xi}\right)^{\frac{1}{\beta}} W^{-\frac{1}{\beta}}\left(B_i\frac{w_i}{P_i}\right)^{\frac{1}{\beta}} \\
\sum_{i=1}^{N} L_i &= \bar{L}.
\end{aligned} \tag{A11}$$

Then we have

$$A_i^{-1} w_i^{1+\theta(1-\xi)} L_i^{1-\theta\psi} t_i^{\theta\xi} = \left(\frac{1}{1-\xi}\right)^{\theta} W^{-\theta}\sum_{n=1}^{N}\tau_{in}^{-\theta} B_n^{\theta} w_n^{1+\theta} L_n^{1-\theta\beta}, \tag{A12}$$

and

$$B_i^{-\theta} w_i^{-\theta} L_i^{\theta\beta} = \left(\frac{1}{1-\xi}\right)^{\theta} W^{-\theta}\sum_{n=1}^{N}\tau_{ni}^{-\theta} A_n L_n^{\theta\psi}\left(w_n^{1-\xi} t_n^{\xi}\right)^{-\theta}. \tag{A13}$$

Then we have

$$A_i^{-1} w_i^{1+\theta(1-\xi)} L_i^{1-\theta\psi} t_i^{\theta\xi} = \phi B_i^{-\theta} w_i^{-\theta} L_i^{\theta\beta}, \tag{A14}$$

where $\phi > 0$ is some scalar.

Therefore,

$$w_i = \phi^{\frac{1}{1+\theta(2-\xi)}} \left(A_i B_i^{-\theta}\right)^{\frac{1}{1+\theta(2-\xi)}} L_i^{\frac{\theta(\beta+\psi)-1}{1+\theta(2-\xi)}} t_i^{-\frac{\theta\xi}{1+\theta(2-\xi)}}. \tag{A15}$$

Then

$$\begin{aligned} & A_i^{-\frac{\theta}{1+\theta(2-\xi)}} B_i^{-\frac{\theta[1+\theta(1-\xi)]}{1+\theta(2-\xi)}} L_i^{\theta\frac{1+[1+\theta(1-\xi)]\beta-\theta\psi}{1+\theta(2-\xi)}} t_i^{\frac{\theta^2\xi}{1+\theta(2-\xi)}} \\ &= \phi^{\frac{1+\theta}{1+\theta(1-\xi)}} \left(\frac{1}{1-\xi}\right)^{\theta} W^{-\theta} \sum_{n=1}^{N} \tau_{in}^{-\theta} A_n^{\frac{1+\theta}{1+\theta(2-\xi)}} B_n^{\frac{\theta^2(1-\xi)}{1+\theta(2-\xi)}} L_n^{\theta\frac{(1-\xi)-\theta(1-\xi)\beta+(1+\theta)\psi}{1+\theta(2-\xi)}} t_n^{-\theta\xi\frac{1+\theta}{1+\theta(2-\xi)}}. \end{aligned} \tag{A16}$$

Let $\breve{\theta} = \frac{\theta}{1+\theta(2-\xi)}$, $\delta_1 = 1 + [1+\theta(1-\xi)]\beta - \theta\psi$, and $\delta_2 = (1-\xi) - \theta(1-\xi)\beta + (1+\theta)\psi$. Then

$$L_i^{\breve{\theta}\delta_1} t_i^{\breve{\theta}\theta\xi} = \phi^{\frac{1+\theta}{1+\theta(1-\xi)}} \left(\frac{1}{1-\xi}\right)^{\theta} W^{-\theta} \sum_{n=1}^{N} K_{in} L_n^{\breve{\theta}\delta_2} t_n^{-\breve{\theta}(1+\theta)\xi}, \tag{A17}$$

where $K_{in} = \tau_{in}^{-\theta} A_n^{\breve{\theta}\left(1+\frac{1}{\theta}\right)} B_n^{\breve{\theta}\theta(1-\xi)} A_i^{\breve{\theta}} B_i^{\breve{\theta}[1+\theta(1-\xi)]}$.

We then log-linearize Equation (A17). For any variables $(Z_i)_{i=1}^N$ with $Z_i > 0$, we denote $\tilde{Z}_i \equiv d\log Z_i$ and the column vector $\tilde{\mathbf{Z}} \equiv \left(\tilde{Z}_1, \ldots, \tilde{Z}_N\right)'$. We also denote $\nabla_{\tilde{\mathbf{t}}}\tilde{\mathbf{Z}} \equiv \left[\frac{\partial \log Z_k}{\partial \log t_i}\right]$ as a matrix with k denoting the rows and i denoting the columns.

Notice that Equation (A17) is derived from (A12). Then we have

$$\tilde{L}_i = -\frac{\theta}{\breve{\theta}\delta_1}\tilde{W} - \frac{\theta\xi}{\delta_1}\tilde{t}_i + \sum_{n=1}^{N} \chi_{in}\left[\frac{\delta_2}{\delta_1}\tilde{L}_n - \frac{(1+\theta)\xi}{\delta_1}\tilde{t}_n\right], \tag{A18}$$

where $\chi_{in} = \frac{X_{in}}{\sum_{k=1}^{N} X_{ik}}$.

We then have the matrix expression as

$$\tilde{\mathbf{L}} = -\frac{\theta}{\breve{\theta}\delta_1}\mathbf{e}\tilde{W} - \frac{\theta\xi}{\delta_1}\tilde{\mathbf{t}} + \frac{\delta_2}{\delta_1}\chi\tilde{\mathbf{L}} - \frac{(1+\theta)\xi}{\delta_1}\chi\tilde{\mathbf{t}}. \tag{A19}$$

Therefore,

$$\tilde{\mathbf{L}} = -\frac{\theta}{\breve{\theta}\delta_1}\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}\tilde{W} - \left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]\tilde{\mathbf{t}}. \tag{A20}$$

We also have

$$\sum_{i=1}^{N}\iota_i\tilde{L}_i = 0. \tag{A21}$$

Therefore

$$0 = \boldsymbol{\iota}'\tilde{\mathbf{L}} = -\frac{\theta}{\breve{\theta}\delta_1}\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}\tilde{W} - \boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]\tilde{\mathbf{t}}. \tag{A22}$$

Then

$$\tilde{W} = -\frac{\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]}{\frac{\theta}{\breve{\theta}\delta_1}\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}}\tilde{\mathbf{t}}. \tag{A23}$$

Then

$$\nabla_{\tilde{\mathbf{t}}}\tilde{W} = -\frac{\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]}{\frac{\theta}{\breve{\theta}\delta_1}\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}}. \tag{A24}$$

The effects of carbon taxes on labor can be expressed as

$$\begin{aligned}\tilde{\mathbf{L}} = {} & \frac{1}{\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}}\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]\tilde{\mathbf{t}} \\ & - \left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]\tilde{\mathbf{t}}.\end{aligned} \tag{A25}$$

Therefore,

$$\tilde{\mathbf{L}} = -\left[\mathbf{I} - \frac{1}{\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}}\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}\boldsymbol{\iota}'\right]\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]\tilde{\mathbf{t}}. \tag{A26}$$

Then

$$\nabla_{\tilde{\mathbf{t}}}\tilde{\mathbf{L}} = -\left[\mathbf{I} - \frac{1}{\boldsymbol{\iota}'\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}}\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\mathbf{e}\boldsymbol{\iota}'\right]\left[\mathbf{I} - \frac{\delta_2}{\delta_1}\chi\right]^{-1}\left[\frac{\theta\xi}{\delta_1}\mathbf{I} + \frac{(1+\theta)\xi}{\delta_1}\chi\right]. \tag{A27}$$

Moreover,

$$\tilde{\mathbf{w}} = \breve{\theta}\left[(\beta+\psi) - \frac{1}{\theta}\right]\tilde{\mathbf{L}} - \breve{\theta}\xi\tilde{\mathbf{t}}. \tag{A28}$$

Since $\tilde{\mathbf{X}} = \tilde{\mathbf{w}} + \tilde{\mathbf{L}}$. Therefore,

$$\nabla_{\tilde{\mathbf{t}}}\tilde{\mathbf{X}} = \left[1 + \breve{\theta}\left(\beta + \psi - \frac{1}{\theta}\right)\right]\nabla_{\tilde{\mathbf{t}}}\tilde{\mathbf{L}} - \breve{\theta}\xi\mathbf{I}. \tag{A29}$$

Q.E.D.